Tutorial1
Tutorial1_Flying

BACK

Questions of today

1. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that f is a polynomial if degree $\leq N$ if and only if f has a removable singularity at infinity and $f(z) \leq M z^N$ for any $z \in \mathbb{C} \setminus \{\}$.
Let $\Omega \subset \mathbb{C}$ be a connected region, show that the set of holomorphic functions in Ω with the usual addition and multiplication of functions, is an integral domain.
2. Show that a continuous function $f : \mathbb{C} \to \mathbb{C}$ holomorphic on $\mathbb{C} \setminus \{0\}$ is entire.
3. **Strengthed version of Cauceuh Weierstrass Theorem:** Suppose f has an essential singularity at $z = 0$ and $z \in \mathbb{C}$ and $z \to 0$. Show that for any $\lambda \in \mathbb{R}$ there is a number $z_\lambda = \rho e^{i\theta}$ such that $f(z) = \lambda$ has infinitely many solutions in $D_\rho(0)$.
4. Let f be holomorphic and ≤ 1 on $D_\rho(0)$. Suppose $|f(0)| = \alpha > 0$. Show that the number of zeroes of f in $D_\rho(0)$ is less than or equal to $\frac{\log \alpha}{\log 2}$.

Hints & solutions of today

1. For the "if" direction, use Cauchy's formula to find the coefficient of the power series expansion of f at the origin.
2. The zero set of a non-zero holomorphic function on a connected domain is discrete
3. Use Weierstra theorem.
(In the tutorial, I mixed up ρ with λ)
4. Use the Cauchy integral formula and Blaschke inequality.

Let $\Omega \subset \mathbb{C}$ be a connected region, show that the set of holomorphic functions $f : \Omega \to \mathbb{C}$ is an integral domain.

Let $z_1, z_2 \in \Omega$ and $f(z_1) \neq 0$. Suppose $f(z_2) = 0$. Since Ω is a connected region, there is a path $\gamma : [a, b] \to \Omega$ such that $\gamma(a) = z_1$ and $\gamma(b) = z_2$. Since f is holomorphic, $f \circ \gamma$ is continuous. Hence, $f \circ \gamma$ is a continuous function on $[a, b]$ and $(f \circ \gamma)(a) \neq 0$ and $(f \circ \gamma)(b) = 0$. Hence, there is a point $t_0 \in [a, b]$ such that $(f \circ \gamma)(t_0) = 0$. Since γ is a path, $\gamma(t_0) = z_0$ for some $z_0 \in \Omega$. Hence, $f(z_0) = 0$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. Show that f is a polynomial if degree $\leq N$ if and only if f has a removable singularity at infinity and $f(z) \leq M z^N$ for any $z \in \mathbb{C} \setminus \{\}